W Web Appendix

This Web Appendix is intended to house the more lengthy proofs of the paper not central to the main results.

W.1 Uniqueness of the high-fee regime subgame equilibrium

Here we show that the uniqueness claims in the equilibrium of the subgame denoted in Lemma 4 i.e., the sellers' equilibrium responses following a fee level $c \in (\underline{c}_2, v]$.[27] Proof that the equilibrium given is an equilibrium in the subgame is given in Appendix A. Given Lemma 1, we need to determine the equilibrium distribution of direct prices p_i^d for $i = 1, 2$, F_1, F_2 to complete the subgame's equilibrium characterization. We denote $\underline{p}_i$ and $\overline{p}_i$ as the min and max of the support of F_i. Firstly, we derive seller responses to $c \in (\underline{c}_2, v]$.

Lemma W1. $\overline{p}_1, \overline{p}_2 \leq v$.

Proof. Any $p_i^d > v$ results in no direct sales and $\pi_i = 0$, whereas $p_i^d = v$ results in direct sales to L_i, giving $\pi_i = vL_i$. □

Lemma W2. $p_1^d, p_2^d \notin (c, v)$.

Proof. Any $p_i^d \in (c, v)$ results in sales to L_i only, such a price is therefore strictly dominated by v. □

Lemma W3. $\underline{p}_1, \underline{p}_2 \geq \underline{c}_2$.

Proof. The best $p_i^d < \underline{c}_2$ can do for i is to sell to shoppers with probability one, netting $\pi_i = p_i^d(\mu + L_i)$, less than the vL_i made from setting $p_i^d = v$. □

Lemma W4. $\min\{\underline{p}_1, \underline{p}_2\} \leq c$.

Proof. Suppose not. Then by Lemmas W1-W2, both sellers play pure strategies $p_i^d = v$. However, as $c > \underline{c}_2$, both sellers have a profitable deviation slightly below c (selling to shoppers and L_i at c netting arbitrarily close to $c(\mu + L_i)$, whereas selling only to L_i at v nets vL_i). □

Lemma W5. $\underline{p}_1 = \underline{p}_2 \equiv \underline{p}$.

Proof. By Lemma W4, $\min\{\underline{p}_1, \underline{p}_2\} \leq c$. Suppose (without loss of generality) that $\underline{p}_1 < \underline{p}_2$. Case 1: $\underline{p}_1 < c$. Prices $p \in [\underline{p}_1, \underline{p}_2)$ are strictly worse than $p \nearrow \underline{p}_2$ for 1. Case 2: $\underline{p}_1 = c$. By Lemmas W1-W2, $p_2^d = v$, giving $\pi_2 = vL_2$. However, as $c > \underline{c}_2$, deviations to $p \nearrow \underline{p}_1$ are profitable for 2. □

Lemma W6. $\underline{p} < c$.

[27] When $c = \underline{c}_2$, the equilibrium of Lemma 4 collapses to that of Lemma 3.

Proof. By Lemmas W4-W5, $\underline{p} \leq c$. Now we rule out $\underline{p} = c$. Case 1: $\underline{p} = c$ and there is a zero probability of a tie at c. Then $c < v$ and both sellers place all their mass on v by Lemmas W1-W2, $\pi_i = vL_i$ but deviating to $p_i^d \in (\underline{c}_2, c)$ nets $\pi_i = p_i^d(\mu + L_i) > vL_i$. Case 2: $\underline{p} = c$ and there is a positive probability of a tie at c. Both sellers have an incentive to shift the mass placed on c to slightly below c (the arbitrarily small loss in price is compensated by the discrete gain in the probability of selling to shoppers). □

Lemma W7. *Direct prices $[\underline{p}, c]$ are in the support of both sellers.*

Proof. Suppose not. By Lemma W5 there is a common element to sellers' supports. Denote $p < c$ highest common element to the supports such that $[\underline{p}, p]$ is in both supports. Suppose without loss of generality that i then has a "hole" in their support of p_i^d, denoted by the interval $(p_l, p_h) \subseteq [\underline{p}, c]$. This cannot be true in equilibrium because for prices close to p_l, j has a profitable deviation of $p_j^d \nearrow p_h$. □

Lemma W8. *No seller has a mass point in $[\underline{p}, c)$.*

Proof. Suppose i had a mass point at $\dot{p} \in [\underline{p}, c)$. By Lemma W7, j's support includes prices in small neighborhoods around $\dot{p}$, where j has a profitable deviation to relocate those p_j^d slightly above $\dot{p}$, to prices slightly below $\dot{p}$ (the arbitrary loss in price is compensated by the discrete gain in the probability of selling to shoppers). □

Lemma W9. *At most one seller has a mass point at c.*

Proof. If both did, then there is a positive probability of a tie at c. Hence, it would be profitable for either to shift the mass they place on c to slightly below c, (the arbitrarily small loss in price is compensated by the discrete gain in the probability of selling to shoppers). [Note that the point the mass is shifted to will not coincide with another mass point: if $\underline{p} = c$, because no prices below $\underline{p}$ are charged; if $\underline{p} < c$, by Lemma W8.] □

Lemma W10. *When $L_1 < L_2$, not both sellers have mass points at v.*

Proof. Suppose so. Case 1: $c < v$. Then $\pi_1 = \underline{p}(\mu + L_1)$ (from Lemmas W7-W8) $= vL_1$ which gives $\underline{p} = \frac{vL_1}{\mu + L_1} \equiv \underline{c}_1 < \underline{c}_2$, ruled out by Lemma W3. Case 2: $c = v$. Lemma W9 applies. □

Lemma W11. *When $c < v$, at least one seller has a mass point at v.*

Proof. Suppose neither seller has a mass point at v. By Lemma W9, at least one seller does not have a mass point at c, call this i. Note that for $j \neq i$, c is in their support by Lemma W7 and $\pi_j = cL_j$, but then j has a profitable deviation $p_j^d = v$, netting vL_j. □

Lemma W12. *When $L_1 < L_2$ and $c = v$, exactly one seller has a mass point at v.*

Proof. By Lemma W10 it is not true that both have mass points at v. Now suppose neither seller has a mass point at v. Then seller $i = 1, 2$ makes $\pi_i = vL_i$, but we know $\pi_i = \underline{p}(\mu + L_i)$ (from Lemmas W7-W8) which gives $\underline{p} = \frac{vL_i}{\mu+L_i} \equiv \underline{c}_i$, which cannot be satisfied for both $i = 1, 2$ because $L_1 < L_2$. □

Lemma W13. *When $L_1 < L_2$, seller 1 has no mass point at v, seller 2 does.*

Proof. When $c < v$, Lemmas W10 and W11 imply that exactly one seller has a mass at v. When $c = v$, Lemma W12 says the same. Next, we show that this seller is seller 2. Suppose instead it was 1, then $\pi_1 = vL_1$. However, 1 has a profitable deviation to $p_1^d = \underline{c}_2$ which generates $\pi_1 = \underline{c}_2(\mu + L_1)$ (the deviation wins shoppers with probability one by Lemmas W3 and W8), which is greater because $L_1 < L_2$. □

Lemma W14. *When $L_1 < L_2$, the unique equilibrium pricing strategies of the subgame starting at $t = 2$ has $p_1 = p_2 = c$, p_1^d and p_2^d mixed over supports $[\underline{c}_2, c]$ and $[\underline{c}_2, c] \cup v$ respectively via the strategies*

$$F_1(p) = \begin{cases} \frac{\mu p - L_2(v-p)}{\mu p} & \text{for } p \in [\underline{c}_2, c) \\ 1 & \text{for } p \geq c, \end{cases}$$

$$F_2(p) = \begin{cases} \frac{\mu p - L_2(v-p)}{\mu p} \frac{\mu+L_1}{\mu+L_2} & \text{for } p \in [\underline{c}_2, c) \\ \frac{\mu c - L_2(v-c)}{\mu c} \frac{\mu+L_1}{\mu+L_2} & \text{for } p \in [c, v) \\ 1 & \text{for } p \geq v. \end{cases}$$

The resulting profits are $\pi_0 = cL_0$, $\pi_1 = \frac{vL_2(L_1+\mu)}{L_2+\mu}$, and $\pi_2 = vL_2$. If $c < v$, $r_0 = 0$; if $c = v$, $r_0 \in [0, 1]$.

Proof. In equilibrium, F_1 must be such that seller 2 is indifferent over all prices in their support. By Lemma W13, 2 has a mass at v, hence $\pi_2 = vL_2$. Therefore, F_1 must satisfy $\pi_2 = p_2^d L_2 + (1 - F_1(p_2^d))\mu p_2^d = vL_2$ for $p_2^d \in [\underline{p}, c]$. Solving gives F_1 over $[\underline{p}, c]$ as stated. By Lemmas W1, W2, W3, W8 and W13, the residual mass of $1 - F_1(c)$ must be located at c. In order for there to be no profitable deviation for 1 to shift this mass slightly below c, $r_0 = 0$. When $c = v$, there is no mass, so any $r_0 \in [0, 1]$ can be supported. Solving $F_1(\underline{p}) = 0$ gives $\underline{p} = \underline{c}_2$. Similarly, F_2 must keep 1 indifferent over all prices in their support. Solving $\pi_1 = \underline{p}(\mu + L_1) = p_1^d L_1 + (1 - F_2(p_1^d))\mu p_1^d$ gives F_2 over $[\underline{p}, c]$ as stated. With the addition of Lemma W9 (to W1, W2, W3, W8 and W13) the residual mass of $1 - F_2(c)$ must be located at v. Checks that the strategies described indeed constitute an equilibrium can be found in the proof of Lemma 4. □

The following Lemmas correspond to the special case of $L_1 = L_2$, which as shown in Lemma W19, does not occur when market composition is endogenously determined. Therefore, we include these results mostly for the sake of completeness.

Lemma W15. *When $L_1 = L_2 \equiv L$ and $c = v$, the unique equilibrium pricing strategies of the subgame starting at $t = 2$ is as given in Lemma W14.*

Proof. By Lemma W9, not both sellers have mass points at v. This means there is at least one seller, i, who makes $\pi_i = vL$ (if neither have a mass at v, $\pi_i = vL$, $i = 1, 2$; if i does, then $\pi_i = vL$). In equilibrium, F_j, $j \neq i$, must be such that seller i is indifferent over all prices in their support: $\pi_i = vL = p_i^d L + (1 - F_j(p_i^d))\mu p_i^d$ for $p_i^d \in [\underline{p}, v]$. Solving for F_j gives the strategy stated in Lemma W14 when $L_1 = L_2 \equiv L$ and $c = v$. There is no residual mass, hence j's strategy features no mass points. Given F_j, we know $\pi_j = \underline{p}(L + \mu) = vL$ and hence i's strategy is dictated by $\pi_j = vL = p_j^d L + (1 - F_i(p_j^d))\mu p_j^d$ for $p_j^d \in [\underline{p}, v]$, which is the same as we had for j. Therefore, the unique equilibrium strategies are symmetric, and are given in Lemma W14 when $L_1 = L_2 \equiv L$ and $c = v$. □

Lemma W16. *When $L_1 = L_2 \equiv L$ (so that $\underline{c}_1 = \underline{c}_2 \equiv \underline{c}$) and $c < v$, equilibria of the subgame starting at $t = 2$ have $p_i = p_j = c$, p_i^d and p_j^d mixed over supports $[\underline{c}, c] \cup v$ and $[\underline{c}, c] \cup v$ respectively for $i, j = 1, 2$ and $i \neq j$ via the strategies*

$$F_i(p) = \begin{cases} \frac{\mu p - L(v-p)}{\mu p} & \text{for } p \in [\underline{c}, c) \\ \alpha & \text{for } p \in [c, v) \\ 1 & \text{for } p \geq v, \end{cases}$$

$$F_j(p) = \begin{cases} \frac{\mu p - L(v-p)}{\mu p} & \text{for } p \in [\underline{c}, c) \\ \frac{\mu c - L(v-c)}{\mu c} & \text{for } p \in [c, v) \\ 1 & \text{for } p \geq v, \end{cases}$$

where $\alpha \in \left[\frac{\mu c - L(v-c)}{\mu c}, 1\right]$. [Note in the case of $\alpha = 1$, v is not in the support of seller i.] The resulting profits are $\pi_0 = cL_0$, $\pi_1 = \pi_2 = vL$. If $\alpha \in \left(\frac{\mu c - L(v-c)}{\mu c}, 1\right]$, $r_0 = 0$; if $\alpha = \frac{\mu c - L(v-c)}{\mu c}$, $r_0 \in [0, 1]$.

Proof. By Lemma W11, at least one seller puts mass on v. This means that at least one makes $\pi_i = vL$ which gives F_j (and $\underline{p}$), over $[\underline{c}, c)$ as stated, in the same way as in proof of Lemma W14. Similarly, $\pi_j = \underline{p}(\mu + L)$ and again the proof of Lemma W14 can be followed to find F_i over $[\underline{c}, c)$ as stated. Both firms have residual mass $1 - F_i(c) = 1 - F_j(c)$ to be allocated. First, and again similar to Lemma W14, by Lemma W9 there can be only one seller with mass on c, hence by Lemmas W1, W2, W3 and W8, there must be one seller (denoted j here) with all this residual mass located at v. However, unlike Lemma W14, there is nothing dictate exactly how firm i should allocate their residual mass $(1 - \alpha)$ between c and v in equilibrium which leaves us with the strategies as stated above. Note that for there to be mass on c, $r_0 = 0$ to a shift of that mass to slightly below c, but if there is none, then any $r_0 \in [0, 1]$ can be supported. Checks similar to those of Lemma 4 confirm these are equilibria. □

W.2 Endogenous Market Composition

This section provides the results allowing for an endogenous market composition $\mathbf{L} = (L_0, L_1, L_2)$ by allowing each firm to determine their stock of loyal consumers. We hold μ constant throughout where the proofs hold for any $\mu > 0$. Note that in the previous sections where $\mathbf{L}$ was exogenous, we adopted the convention of naming the direct channels such that $L_1 \leq L_2$, but now that order of the labels should be discarded: There are simply three firms, of any size, indexed $i = 0, 1, 2$ where 0 is a competitive channel and $1, 2$ are direct channels. Formally, we add a stage, $t = 0$ to the game where each player $i = 0, 1, 2$ simultaneously chooses L_i subject to a C^2 cost function $\psi_i : \mathbb{R}^3_+ \to \mathbb{R}_+$. The cost to player i of choosing L_i given the other players choose L_{-i} is written $\psi_i(L_i|L_{-i})$ where $L_{-i} = \mathbf{L} \setminus L_i$. We make the following additional assumptions on the cost function for $i, j = 0, 1, 2$:

1. Symmetry: $\psi_i = \psi_j \equiv \psi$
2. Costs nothing to do nothing: $\psi(0|\cdot) = 0$
3. Increasing: $\frac{\partial \psi(L_i|L_{-i})}{\partial L_i} = 0$ for $L_i = 0$; $\frac{\partial \psi}{\partial L_i} > 0$ for $L_i > 0$
4. Convex: $\frac{\partial^2 \psi(L_i|L_{-i})}{\partial L_i^2} > 0$
5. Inter-dependency: $\frac{\partial \psi(L_i|L_{-i})}{\partial L_j} = 0$ for $L_i = 0$; $\frac{\partial \psi(L_i|L_{-i})}{\partial L_j} > 0$ for $L_i > 0$

Players are assumed to have access to the same marketing tools, hence ψ is common across players (assumption 1). Assumptions 3 and 4 state respectively that determining market power is costly, and increasingly so. Assumption 5 states that it becomes more costly to generate the same market power when other players are marketing more.

Denote low-fee regime, LFR, and high-fee regime, HFR. We view this marketing stage of the game as reflecting long-run decisions and therefore restrict attention to pure-strategy equilibria. Denote a cost function satisfying assumptions 1-5 as ψ, and the set of all these functions Ψ. Let Ψ_L (Ψ_H) denote the set of all functions ψ under which a low-fee regime (high-fee regime) equilibrium exists.

Lemma W17. *In any equilibrium, $L_i^* > 0$ for all i.*

Proof. Suppose $L_i^* = 0$ for some i. Equilibrium profit for i is weakly positive and the marginal cost of L_i at $L_i = 0$ is zero. Therefore, if the marginal benefit of L_i at $L_i = 0$ is strictly positive, there is some strictly profitable deviation away from $L_i^* = 0$.

If $i = 0$, the marginal benefit at $L_i = 0$ is $\frac{d(vL_i)}{dL_i} = v > 0$ if the HFR results; and $v\frac{L_1^*}{L_1^*+\mu} > 0$ if the LFR results (note the LFR implies $L_1^* > 0$). If $i > 0$, then $\hat{L}_i > 0$ small produces the LFR with a marginal benefit of $v\frac{L_2^*}{L_2^*+\mu} > 0$ if $\hat{L}_i \leq L_j^*$ for all $j > 0$ or $v > 0$ otherwise. □

Lemma W18. *No symmetric equilibrium exists.*

Proof. Suppose instead that $L_i = L^*$ for all i. By Lemma W17, $L^* > 0$. A deviation of any player to $\hat{L}_i$ yields profit:

$$\hat{\pi}(\hat{L}_i) = \begin{cases} v\frac{L^*}{L^*+\mu}(\hat{L}_i + \mu) - \psi(\hat{L}_i|\mathbf{L}^*_{-i}) & \text{for } \hat{L}_i \leq L^* \\ v\hat{L}_i - \psi(\hat{L}_i|\mathbf{L}^*_{-i}) & \text{for } \hat{L}_i \geq L^*. \end{cases}$$

[Notice this is regardless of whether the LFR or HFR results in equilibrium.] The function $\hat{\pi}$ is continuous, but non-differentiable at $\hat{L}_i = L^*$. To see this, note that the left and right derivatives at $\hat{L}_i = L^*$ are $\partial_-\hat{\pi}(L^*) = v\frac{L^*}{L^*+\mu} - \frac{\partial\psi(L_i|\mathbf{L}_{-i})}{\partial L_i}\Big|_{\mathbf{L}=\mathbf{L}^*}$ and $\partial_+\hat{\pi}(L^*) = v - \frac{\partial\psi(L_i|\mathbf{L}_{-i})}{\partial L_i}\Big|_{\mathbf{L}=\mathbf{L}^*}$ respectively, hence $\partial_- < \partial_+$. However, for there to be no strictly profitable deviation slightly below or above L^* we need $\partial_- \geq 0$ and $\partial_+ \leq 0$ which implies $\partial_- \geq \partial_+$, a contradiction. □

Lemma W19. *If the LFR obtains in equilibrium, $L_0^* < L_1^*$. If the HFR obtains in equilibrium, $L_1^* < L_2^*$.*

Proof. If instead $L_0^* = L_1^*$, the profit from deviations of the CC, $\hat{L}_0$ is given by:

$$\hat{\pi}(\hat{L}_0) = \begin{cases} v\frac{L_1^*}{L_1^*+\mu}(\hat{L}_0 + \mu) - \psi(\hat{L}_0|\mathbf{L}^*_{-0}) & \text{for } \hat{L}_0 \leq L_0^* \\ v\hat{L}_0 - \psi(\hat{L}_0|\mathbf{L}^*_{-0}) & \text{for } \hat{L}_0 \geq L_0^*. \end{cases}$$

Following the proof of Lemma W18 shows that such an equilibrium implies a contradiction. Similarly, if $L_1^* = L_2^* \equiv L^* < L_0^*$, the profit from deviations by $i = 1, 2$, $\hat{L}_i$ is given by:

$$\hat{\pi}(\hat{L}_i) = \begin{cases} v\frac{L^*}{L^*+\mu}(\hat{L}_i + \mu) - \psi(\hat{L}_i|\mathbf{L}^*_{-i}) & \text{for } \hat{L}_i \leq L^* \\ v\hat{L}_i - \psi(\hat{L}_i|\mathbf{L}^*_{-i}) & \text{for } \hat{L}_i \geq L^*. \end{cases}$$

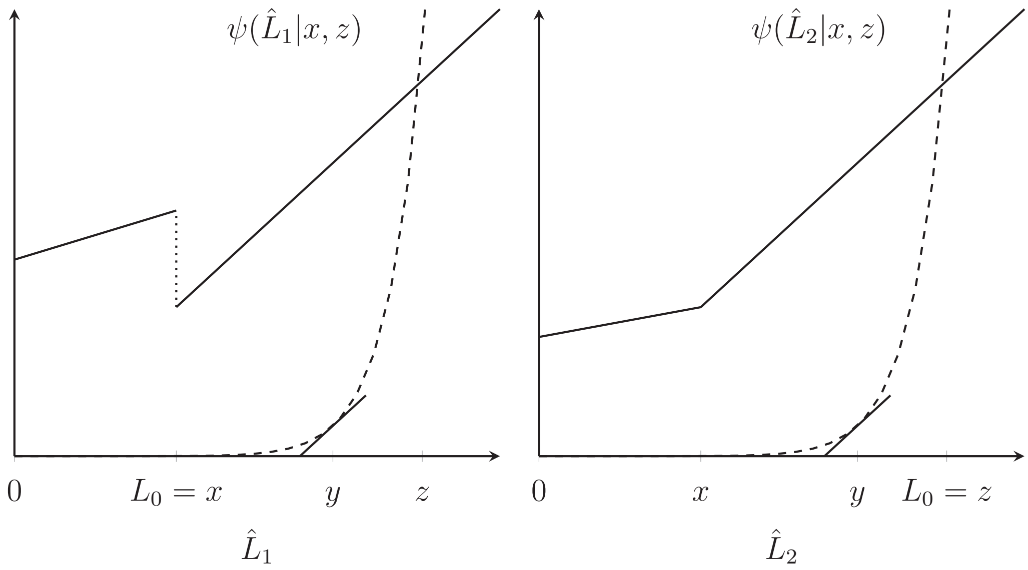
Following the proof of Lemma W18 shows that such an equilibrium implies a contradiction. □

Proposition 8 (High-fee regime predominance). *There are more cost functions that lead to a high-fee regime than a low-fee regime in equilibrium, i.e., $|\Psi_H| \geq |\Psi_L|$.*

Proof. We show that $\psi \in \Psi_L \implies \psi \in \Psi_H$ i.e., for any ψ that supports an equilibrium which gives the low-fee regime, then under the same cost function there is also an equilibrium which gives the high-fee regime.

Suppose $\psi \in \Psi_L$ and denote an equilibrium which gives the low-fee regime as $(L_0, L_1, L_2) = (x, y, z)$ where $x \leq y \leq z$. We show that this implies that there also exists an equilibrium where firms choose (z, x, y) which gives the high-fee regime. By Lemmas W19 and

W17, $0 < x < y$ for (x, y, z) to be an equilibrium. Below on the left is an illustration of such a low-fee equilibrium (x, y, z). On the right is an illustration of the corresponding high-fee equilibrium strategies $(L_0, L_1, L_2) = (z, x, y)$. The x-axis represents unilateral deviations of the player choosing y (seller in the left-hand panel and seller 2 in the right-hand panel). [To prevent clutter, the panels show only the revenue and cost curves of the players choosing y, but the procedure is unchanged if done instead for the players choosing x or z.]



Left panel: Deviation incentives for the smaller seller from the low-fee regime equilibrium (x, y, z).
Right panel: Deviation incentives for the larger seller from the high-fee regime equilibrium (z, x, y).

As (x, y, z) is an equilibrium, there are no profitable deviations for the players either (i) locally: the slope of the cost curve is equal to the slope of the revenue curve; or (ii) globally: there is no profitable deviation to, or slightly below the choice of the smallest player. It can now be seen that (z, x, y) is also an equilibrium. Because the same three values x, y, z are chosen (albeit by different players) in (x, y, z) and (z, x, y), we show that (z, x, y) satisfies (i) and (ii), which implies it is an equilibrium. That it is a high-fee regime equilibrium is immediate because in (z, x, y), $L_0 > L_1$. To see this for the player choosing y, notice that the figures in both panels are identical except for the revenue curve which is the same for deviations above x, but strictly higher in the left-hand panel for deviations below x. [Although only the deviations for the player at y are shown, it is also true that the revenue curve in the left panel is weakly above the revenue curve in the right panel for the players at x and z, hence the argument can be repeated for all players.] This implies immediately that there are no local deviations. It also shows that if a deviation to at or slightly below x is not profitable in the left panel, it is not profitable for the player in the right panel, which shows there are no global deviations. Hence if (x, y, z) is an equilibrium, then (z, x, y) is too. □

W.3 Commission types and a marginal cost of production

Through its selection of fees (flat or ad valorem), the CC determines the equilibrium price available through it, p. As such, in this section it is convenient to write as if the CC is directly determining p. We state the equilibria in terms of p in the same order as in our presentation of the baseline model. In the baseline model, $\underline{c}_i$ was the threshold fee level such that seller i would prefer to sell to shoppers and their loyals direct, but at the CC price, and sell only to their loyals direct at the monopoly price. The corresponding expressions in terms of p, while allowing for a marginal cost of production, $m \in [0, v]$, are given below for $i = 1, 2$.

$$\underline{p}_i = \frac{vL_i + m\mu}{L_i + \mu}$$

Lemma W20 (Low fee regime). *Suppose $0 \leq p \leq \underline{p}_1$. An equilibrium of the subgame starting at $t = 2$ has $p_1 = p_2 = p$, $p_1^d = p_2^d = v$ and any $r_0 \in [0, 1]$. The resulting equilibrium profits are $\pi_0 = (p\tau + c)(L_0 + \mu) = (p - m)(L_0 + \mu)$,*[28] *$\pi_1 = (v - m)L_1$, $\pi_2 = (v - m)L_2$. When $0 \leq p < \underline{p}_1$, this equilibrium is unique.*

Lemma W21 (Mid fee regime). *Suppose $\underline{p}_1 \leq p \leq \underline{p}_2$. An equilibrium of the subgame starting at $t = 2$ has $p_1 = p_2 = p$, $p_1^d = p$, $p_2^d = v$ and $r_0 = 0$. The resulting equilibrium profits are $\pi_0 = (p-m)L_0$, $\pi_1 = (p-m)(L_1+\mu)$, and $\pi_2 = (v-m)L_2$. When $\underline{p}_1 < p < \underline{p}_2$, this equilibrium is unique.*

Lemma W22 (High fee regime). *Suppose $\underline{p}_2 \leq p \leq v$. An equilibrium of the subgame starting at $t = 2$ has $p_1 = p_2 = p$, p_1^d and p_2^d mixed over supports $[\underline{p}, p]$ and $[\underline{p}, p] \cup v$ respectively via the strategies*

$$F_1(p^d) = \begin{cases} 1 - \frac{(v-p^d)L_2}{(p^d-m)\mu} & \text{for } p^d \in [\underline{p}, p) \\ 1 & \text{for } p^d \geq p, \end{cases}$$

$$F_2(p^d) = \begin{cases} 1 - \frac{(p-m)\mu-(p^d-\underline{p})L_1}{(p^d-m)\mu} & \text{for } p^d \in [\underline{p}, p) \\ \frac{(p-m)\mu-(p-\underline{p})L_1}{(p-m)\mu} & \text{for } p^d \in [p, v) \\ 1 & \text{for } p^d \geq v, \end{cases}$$

where $\underline{p} = \underline{p}_2$ and $r_0 = 0$. The resulting profits are $\pi_0 = (p - m)L_0$, $\pi_1 = \frac{(v-m)L_2(L_1+\mu)}{L_2+\mu}$, and $\pi_2 = (v - m)L_2$. When $p = v$, any $r_0 \in [0, 1]$ can be supported. When $\underline{p}_2 < p \leq p$, this equilibrium is unique.

Proposition W1. *When $L_0 \leq L_1$, there is a low-fee equilibrium: the competitive channel sets $p = \underline{p}_1$ and sellers price in accordance with Lemma 2. When $L_0 \geq L_1$, there is a*

[28] Here and below we use that fact that $p(1 - \tau) - c - m = 0$ because firms compete away all profit on the CC.

high-fee equilibrium: the competitive channel sets $p = v$ and sellers price in accordance with Lemma 4.

W.4 Commission discrimination

Proposition W2. *Suppose the CC can charge fee c_1 to seller 1 and c_2 to seller 2. It will choose $c_1 = c_2$ in any equilibrium.*

Proof. If the CC offers fees $c_i < c_j \leq v$ Bertrand competition on the CC implies that all sales via the CC go to firm i at a price of $p_i = c_j$.

Part (i): Suppose that $\min\{p_1^d, p_2^d\} \leq \min\{p_1, p_2\}$ with probability 1 in the pricing sub-game and $c_i < c_j \leq v$. Then the CC's profit is $\pi_0 = c_i L_0$.[29] There is a profitable deviation to $c_1 = c_2 = v$.

Part (ii): Suppose that $\min\{p_1^d, p_2^d\} \leq \min\{p_1, p_2\}$ with probability 0 and $c_i < c_j \leq v$. We show that the CC can serve the same mass of consumers at a higher fee by reducing c_j. Indeed, firm i earns profit of $vL_i + (c_j - c_i)(L_0 + \mu)$ (assuming it sets $p_i^d = v$, which is optimal conditional on not undercutting the CC). The best deviation would be to $p_i^d = c_j$ to undercut the CC, yielding profit $c_j(L_i + \mu) + (c_j - c_i)L_0$. The deviation is not profitable if $c_i \leq \widetilde{c}_i \equiv \frac{L_i(v-c_j)}{\mu}$. Similarly, firm j earns vL_j. A deviation to $p_j^d = c_j$ yields profit $c_j(L_j + \mu)$ and is not profitable if $c_j \leq \underline{c}_j$. Thus, the best that the CC can do, conditional on deterring undercutting, is to solve

$$\max_{c_i, c_j} c_i(\mu + L_0)$$

$$\text{s.t. } c_i \leq c_j, c_i \leq \widetilde{c}_i, c_j \leq \underline{c}_j.$$

At least one of $c_i \leq \widetilde{c}_i$ and $c_j \leq \underline{c}_j$ must bind and both are made more slack by a reduction in c_j.

Part (iii) Lastly, if $c_i < c_j$ we show that it can't be the case that $\min\{p_1^d, p_2^d\} \leq \min\{p_1, p_2\}$ with probability in $(0, 1)$. Indeed, this implies that both firms put positive mass on direct prices below c_j and positive mass on prices above. Direct prices in (c_j, v) never serve shoppers and are dominated by v. Standard arguments then imply that both firms must share a support, $[\underline{p}, c_j] \cup \{v\}$. To be indifferent between $p_j^d = v$ and $p_j^d = \underline{p}$, j must have $\underline{p}(L_j + \mu) = vL_j \iff \underline{p} = \underline{c}_j$, which does not depend on c_i or c_j. Similarly, i is indifferent if $\underline{p} = \frac{L_i v + \mu(c_j - c_i)}{L_i + \mu}$. For the two values of $\underline{p}$ to conincide, we require $(L_j - L_i)v = (L_j + \mu)(c_j - c_i)$, which implies that $L_j > L_i$ (i.e., $j = 2$ and $i = 1$).

To be indifferent between $\underline{p}$ and a $p_2^d < c_2$, 2 must have $\underline{p}(L_2 + \mu) + L_0(c_2 - c_1) =$

[29] If $\min\{p_1^d, p_2^d\} = \min\{p_1, p_2\}$ then the CC's profit is $c_i(L_0 + r_0\mu)$. But we must have $r_0 = 0$ in such an equilibrium or else a firm would want to reduce p_i^d to break the tie.

$p_2^d L_2 + L_0(c_2 - c_1) + \mu p_i^d(1 - F_1(p_2^d))$, i.e.

$$F_1(p_2^d) = \frac{(L_2+\mu)(p_2^d - \underline{p})}{p_2^d \mu} = \frac{(L_2+\mu)(p_2^d - \underline{c}_2)}{p_2^d \mu}.$$

Similarly, 1 is indifferent if

$$F_2(p_1^d) = \frac{(L_1+\mu)(p_1^d - \underline{p})}{p_1^d \mu} = \frac{(L_1+\mu)(p_1^d - \underline{c}_2)}{p_1^d \mu}.$$

The CC's profit is $\pi_0 = c_1[1 - F_1(c_2)][1 - F_2(c_2)]\mu + c_1 L_0$.[30] Observe that F_1 and F_2 do not depend on c_1, so π_0 is linear in c_1 and maximized when $c_1 \uparrow c_2$. Letting $c_1 = c_2 = c$, it is easily veriefied that π_0 is convex and maximized at either $c = v$ (inducing firms to undercut with probability 1) or at $c = \underline{c}_1$ (inducing firms to undercut with probability zero). Profit when $c_1 \neq c_2$ must be strictly less so that the CC has a profitable deviation to the case considered in either part (i) or part (ii). □

W.5 Search and Information Frictions

Consumers only aware of firm 1 or 2 (I_1, I_2 such that $I_2 \geq I_1 > 0$) act as loyals in the baseline. Consumers aware of the CC, firm 0 ($I_0 > 0$), become aware of firm 1 and 2 when they see them on the CC, and then make a rational decision of whether to check firms' direct prices, subject to marginal search costs (they do not otherwise observe firms direct prices, but they have correct expectations in equilibrium). As in much of the search literature, we assume consumers' first search is free, hence consumers I_1 and I_2 incur no search cost, and I_0 incur no search cost for their visit to the CC, but any subsequent visit to firms is costly. The first and such such visit costs σ_1 and σ_2, respectively, where $\sigma_1 \leq \sigma_2$. Shoppers are aware of all prices (they incur no search or information costs). The I_0 consumers make their search decisions simultaneously, based on firms' equilibrium price distributions. The game is otherwise unchanged. We proceed by considering equilibria in the subgames following the CC's choice of commission, c. Our first observation is that when $c \in [0, vI_2/(I_2+\mu)]$ the strategies of Lemmas 2 and 3, are also equilibrium strategies in our search model, for any $0 \leq \sigma_1 \leq \sigma_2$. The details are given in the Lemmas below. In this section, all the equilibria we report feature $r_0 = 0$ so we do not report it when stating the results.

Lemma W23 (Low-fee regime). *Suppose $0 \leq c \leq vI_1/(I_1+\mu)$ and $0 \leq \sigma_1 \leq \sigma_2$. An equilibrium of the subgame starting at $t = 2$ has $p_1 = p_2 = c$, $p_1^d = p_2^d = v$, and consumers informed of the CC search only the CC. Profits are $\pi_0 = c(I_0 + \mu)$, $\pi_1 = vI_1$, $\pi_2 = vI_2$.*

[30] A tie between CC and direct prices must be broken in the direct channel's favor, otherwise a seller would wish to undercut the tie with its direct price.

Lemma W24 (Mid-fee regime). *Suppose* $vI_1/(I_1+\mu) \le c \le vI_2/(I_2+\mu)$ *and* $0 \le \sigma_1 \le \sigma_2$*. An equilibrium of the subgame starting at* $t = 2$ *has* $p_1 = p_2 = c$*,* $p_1^d = c$*,* $p_2^d = v$*, and consumers informed of the CC search only the CC. Profits are* $\pi_0 = cI_0$*,* $\pi_1 = c(I_1 + \mu)$*, and* $\pi_2 = vI_2$*.*

In the low-fee and mid-fee regimes, the lowest prices are hosted on the CC. Therefore, I_0 consumers prefer not to search (weakly if $\sigma_1 = 0$). Given they do not search, firms have no incentive to deviate from their strategies, and the deviation checks in the baseline model apply. For higher levels of c, the baseline's high-fee regime strategies continue to be equilibrium strategies when search is sufficiently costly. To express our results we define the following distribution functions.

$$A_1(p; s_2) = \begin{cases} \frac{\mu p - (I_2 + s_2 I_0)(v-p)}{\mu p} & \text{for } p \in \left[\frac{v(I_2+s_2I_0)}{\mu+I_2+s_2I_0}, c\right) \\ 1 & \text{for } p \ge c, \end{cases}$$

$$A_2(p; s_1, s_2) = \begin{cases} \frac{\mu p - (I_2 + s_2 I_0)(v-p)}{\mu p} \frac{\mu+I_1+s_1I_0}{\mu+I_2+s_2I_0} & \text{for } p \in \left[\frac{v(I_2+s_2I_0)}{\mu+I_2+s_2I_0}, c\right) \\ \frac{\mu c - (I_2 + s_2 I_0)(v-c)}{\mu c} \frac{\mu+I_1+s_1I_0}{\mu+I_2+s_2I_0} & \text{for } p \in [c, v) \\ 1 & \text{for } p \ge v \end{cases}$$

$$A_S(p; s_2) = \begin{cases} \frac{(\mu+s_2I_0)p - I_2(v-p)}{\mu p} & \text{for } p \in \left[\frac{vI_2}{\mu+I_2+s_2I_0}, c\right) \\ \frac{(\mu+s_2I_0)c - I_2(v-p)}{\mu c} & \text{for } p \in [c, v) \\ 1 & \text{for } p \ge v. \end{cases}$$

$$B_S(p; s_2) = \begin{cases} \frac{\mu p - (I_2 + s_2 I_0)(c-p)}{\mu p} & \text{for } p \in \left[\frac{c(I_2+s_2I_0)}{\mu+I_2+s_2I_0}, c\right) \\ 1 & \text{for } p \ge c. \end{cases}$$

Let $\mathbb{E}_X[\mathbf{z}]$ denote the expected price from distribution X at parameter values $\mathbf{z}$, and $\mathbb{E}_{X,Y}[\mathbf{z}]$ denote the expected minimum of one draw from each of X and Y, both evaluated at parameter values $\mathbf{z}$.

Lemma W25 (High-fee regime; high search costs). *Suppose* $vI_2/(I_2 + \mu) < c \le \min\{\sigma_1 + \mathbb{E}_{A_1}[0], v\}$ *and* $0 \le \sigma_1 \le \sigma_2$*. An equilibrium of the subgame starting at* $t = 2$ *has* $p_1 = p_2 = c$*,* $p_1^d \sim A_1(\cdot; 0)$*,* $p_2^d \sim A_2(\cdot; 0, 0)$*, and consumers informed of the CC search only the CC. Profits are* $\pi_0 = cI_0$*,* $\pi_1 = vI_2(I_1 + \mu)/(I_2 + \mu)$*, and* $\pi_2 = vI_2$*.*

The equilibrium benefit from searching the smaller firm when there is no search is $c - \mathbb{E}_{A_1}[0]$. Where consumers search one firm, they search firm 1 because $\mathbb{E}_{A_1}[0] < \mathbb{E}_{A_2}[0, 0]$. Note that if it is unprofitable to search firm 1, then it is unprofitable to search both 1 and 2 because the marginal search cost is weakly increasing, but the marginal expected benefit of the second search is less than the first: $c - \mathbb{E}_{A_1}[0] > \mathbb{E}_{A_1}[0] - \mathbb{E}_{A_1,A_2}[0, 0]$.

For lower levels of σ_1, high-fee regime equilibria of the subgame starting in $t = 2$ can feature search. Below we characterize an equilibrium when $\sigma_1 = \sigma_2 = 0$ and I_0 consumers choose to search the CC and both firms.

Lemma W26 (High-fee regime; zero search costs). *Suppose $vI_2/(I_2 + \mu) < c \leq v$ and $\sigma_1 = \sigma_2 = 0$. An equilibrium of the subgame starting at $t = 2$ has $p_1 = p_2 = c$, $p_1^d \sim A_1(\cdot; 0)$, $p_2^d \sim A_2(\cdot; 0, 0)$ where μ is replaced by $\mu + I_0$. Consumers informed of the CC also visit both firms. Profits are $\pi_0 = 0$, $\pi_1 = vI_2(I_0 + I_1 + \mu)/(I_2 + \mu)$, and $\pi_2 = vI_2$.*

Lemma W27 (Market equilibrium with zero search costs). *When $I_0 < I_1(I_2 + \mu)/(I_2 - I_1)$, the low-fee equilibrium results: the competitive channel sets $c = vI_1/(I_1 + \mu)$, sellers price and consumers search in accordance with Lemma W23. When $I_0 > I_1(I_2 + \mu)/(I_2 - I_1)$, the mid-fee equilibrium results: the competitive channel sets $c = v$, sellers price and consumers search in accordance with Lemma W24. When $I_0 = I_1(I_2 + \mu)/(I_2 - I_1)$, both equilibria exist.*

We have derived equilibria with no search and full search. We now derive equilibria with partial search. We assume that $\sigma_1 \geq 0$ while searching twice is prohibitively costly e.g., $\sigma_2 = \infty$. Let s_1 and s_2 denote the share of the I_0 consumers who search firm 1 and 2, respectively. Firstly, note that when $vI_2/(I_2 + \mu) < c \leq \min\{\sigma_1 + \mathbb{E}_{A_1}[0], v\}$, Lemma W25 applies and $s_1 = s_2 = 0$. However, for higher levels of c, there is partial search in equilibrium:

Lemma W28 (High-fee regime; intermediate search costs & partial search). *Suppose $\sigma_1 + \mathbb{E}_{A_1}[0] < c \leq v$, $\sigma_1 \geq 0$ and $\sigma_2 = \infty$. Define $R = I_0 - I_2 + I_1$. There exist the following equilibria in the subgames starting at $t = 2$:*

1. *$R \leq 0$. $p_1 = p_2 = c$, $p_1^d \sim A_1(\cdot; 0)$, $p_2^d \sim A_2(\cdot; 1, 0)$. In $t = 3$, $s_1 = 1$ and $s_2 = 0$. Profits are $\pi_0 = 0$, $\pi_1 = vI_2(I_0 + I_1 + \mu)/(I_2 + \mu)$, and $\pi_2 = vI_2$.*

2. *$R \in (0, 2\mu)$.*
   - *$c \in \left[\frac{vI_2}{\mu + I_2}, \frac{vI_2}{\mu + R/2}\right]$. $p_1 = p_2 = c$, $p_1^d, p_2^d \sim A_S(\cdot; s_2)$, $s_1 = (2I_0 - R)/(2I_0)$ and $s_2 = R/(2I_0)$. The resulting profits are $\pi_0 = 0$, $\pi_1 = vI_2$, and $\pi_2 = vI_2$.*
   - *$c \in \left[\frac{vI_2}{\mu + R/2}, v\right]$. $p_1 = p_2 = c$, $p_1^d, p_2^d \sim B_S(\cdot; s_2)$, $s_1 = (I_2 - I_1)/I_0 + s_2$, where $s_2 = \min\{\tilde{s}, R/(2I_0)\}$ and $\tilde{s}$ is the unique solution to*

$$c - \mathbb{E}_{B_S}[\tilde{s}] = \sigma_1. \tag{9}$$

   Profits are $\pi_0 = c(1 - s_1 - s_2)$, $\pi_1 = c(I_2 + s_2 I_0)$, and $\pi_2 = c(I_2 + s_2 I_0)$.

3. *$R \geq 2\mu$. Same as $R \in (0, 2\mu)$ in the case of $c \in \left[\frac{vI_2}{\mu + R/2}, v\right]$.*

Proof. Start in the scenario of Lemma W25 where $s_1 = s_2 = 0$, and consider exogenously increasing search of one firm. At some point, $\sigma_1 + \mathbb{E}_{A_1}[0] < c$ i.e., consumers strictly prefer to search firm 1 than not to search (because $\mathbb{E}_{A_1}[0] < \mathbb{E}_{A_2}[0, 0]$, consumers search firm 1 not firm 2). Therefore, consider s_1 increasing.

1. Can we have an equilibrium with firms mixing via A_1, A_2 in direct prices? Distribution A_1 is independent of s_1, and hence if consumers strictly prefer to search, they all search. The question is whether A_1 is valid when $s_1 = 1$. A_1 is only valid if firm 1 remains weakly smaller than firm 2 after search i.e., $I_1 + s_1 I_0 \leq I_2 + s_2 I_0$, which is $I_1 + s_1 I_0 \leq I_2$ because $s_2 = 0$ when A_1 applies. If $R = I_0 - I_2 + I_1 \leq 0$ then we can have an equilibrium where $s_1 = 1$ and A_1 applies because $I_1 + I_0 \leq I_2$. This covers the first bullet.

2. What if $R \geq 0$? The equilibrium of the previous point cannot continue: consumers cannot all search firm 1 because that would make firm 1 larger than firm 2, which would make the expected price of firm 2 more attractive which would contradict consumers searching firm 1. Therefore, once there is sufficient search to make firms of equal size, any further search must be equally split across firms 1 and 2 such that they are of equal size i.e., $I_1 + s_1 I_0 = I_2 + s_2 I_0$, or $s_1 = (I_2 + s_2 I_0 - I_1)/I_0$. For this to be consistent with equilibrium search behavior, firms must price via identical distributions, which is achieved by A_S, which is also a symmetric mutual best response for firms. A_S is only valid if the mass point exists $A_S(c; s_2) \geq 0 \iff c \leq vI_2/(s_2 I_0 + I_2)$. A_S has the feature that $\partial E_{A_S}[s_2]/\partial s_2 < 0$, and therefore all consumers search in any equilibrium with A_S i.e., $s_1 + s_2 = 1$, and $s_2 = R/(2I_0)$. A_S is valid if $c \leq vI_2/(R/2 + I_2)$. Because $c > vI_2/(\mu + I_2)$, this equilibrium applies only if additionally, $R < 2\mu$.

3. What if we follow the logic in the case of $R \geq 0$ above, but $R \geq 2\mu$? A_S is no longer valid, but B_S is, and is a mutual best response for firms. In contrast to A_S, $\partial E_{B_S}[s_2]/\partial s_2 < 0$ so that equilibria with partial search are possible. Whether or not there is full or partial search depends on how many consumers there are left to search beyond the number required to make the firms equally sized. If this number is small, there is full search; if large, partial search. The relevant threshold is the level of search required beyond that to make the firms equally sized such that consumers are indifferent to search, $\tilde{s}$:

$$c - \mathbb{E}_{B_S}[\tilde{s}] = \sigma_1,$$

   which has a unique solution. If $\tilde{s} \geq R/2$, the level of search required to make consumers indifferent exceeds the number of consumers available and so there is full search, $s_1 + s_2 = 1$. If $\tilde{s} < R/2$, there is partial search, $s_1 + s_2 < 1$ where $s_1 = (I_2 - I_1)/I_0 + \tilde{s}$ and $s_2 = \tilde{s}$.

□

Unlike the case of zero search costs, the CC optimally chooses a fee beyond $vI_2/(I_2+\mu)$ within the high-fee regime because consumers face search costs which deter search at high price levels. This makes the high-fee regime more profitable. Naturally, the CC does (weakly) better when search costs are higher. Below we report an equilibrium of the game as a whole which features an interior CC-commission level, partial consumer search and direct prices that are sometimes above and sometimes below CC prices.

Lemma W29 (Market equilibrium with intermediate search costs). *Suppose* $0 < \sigma_1 < v - \mathbb{E}_{A_1}[0]$, $\sigma_2 = \infty$ *and* $I_0 < I_2 - I_1$. *Define* $\tilde{c}$ *as the unique solution to* $\sigma_1 = c - \mathbb{E}_{A_1}[0]$.[31] *When* $I_0 < v\mu I_1/(\tilde{c}(I_1+\mu) - vI_1)$, *the low-fee equilibrium results: the competitive channel sets* $c = vI_1/(I_1+\mu)$, *sellers price and consumers search in accordance with Lemma W23. When* $I_0 > v\mu I_1/(\tilde{c}(I_1+\mu) - vI_1)$, *the high-fee equilibrium results: the competitive channel sets* $c = \tilde{c} \in (vI_2/(I_2+\mu), v)$, *sellers price and consumers search in accordance with Lemma W28. When* $I_0 = v\mu I_1/(\tilde{c}(I_1+\mu) - vI_1)$, *both equilibria exist.*

[31] Note that $\tilde{c}$ is independent of I_0.